Shear free solutions in General Relativity Theory

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Abstract

The Goldberg-Sachs theorem is an exact result on shear-free null geodesics in a vacuum spacetime. It is compared and contrasted with an exact result for pressure-free matter: shear-free flows cannot both expand and rotate. In both cases, the shear-free condition restricts the way distant matter can influence the local gravitational field. This leads to intriguing discontinuities in the relation of the General Relativity solutions to Newtonian solutions in the timelike case, and of the full theory to the linearised theory in the null case.

It is a pleasure to dedicate this paper to Josh Goldberg.

Key words: General Relativity, Exact solutions, Shear-free fluid flows, Shear free null rays, Goldberg-Sachs Theorem.

1 Introduction

The physical interpretation of solutions of the Einstein Field Equations in General Relativity Theory is intimately tied in to the way timelike and null geodesics behave. The differential properties of families of geodesics are described by their expansion, rotation and shear in the timelike case [12, 15], and by the null expansion and shear in the case of the irrotational null geodesic congruences that underlie observations [14]. This paper discusses the key role of shear in physical processes as evidenced by their effect on such congruences, and hence the very special nature of shear-free solutions. The remarkable Goldberg-Sachs theorem [20] demonstrates this very special nature in the case of shear free null geodesics. It was proceeded by Gödel's intriguing results shear-free timelike geodesics [19], which considered specific the case of spatially homogeneous geometries. This result was generalized to the inhomogeneous case of any shearfree timelike geodesics in [16]. Although very different in detail, the timelike and null cases are in a sense analogous results: they both refer to the way that shear in a congruence conveys information about distant matter, so shearfree congruences can only occur in restricted circumstances. That is what is explored in this paper.

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The Einstein Field Equations ('EFE') take the form

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} = \kappa T_{ab} - \Lambda\, g_{ab}\ , \tag{1}$$

showing how matter causes space time curvature by specifying the spacetime Ricci tensor R_{ab} in terms of the matter stress tensor T_{ab}. Provided the cosmological constant Λ is constant in time and space, the twice-contracted Bianchi identities together with (1) guarantee the conservation of total energy-momentum:

$$\{\nabla_b G^{ab} = 0,\ \ \nabla_a \Lambda = 0\} \quad\ \Leftrightarrow \quad\ \nabla_b T^{ab} = 0. \tag{2}$$

To complete the dynamical description, we must specify the matter present by providing suitable equations of state relating the components of T_{ab}.

The locally free gravitational field is given by Weyl tensor C_{abcd}, which is the trace-free part of the full curvature tensor R_{abcd}. It is the part of the spacetime curvature not directly determined pointwise by matter but rather determined by matter elsewhere through tidal effects and gravitational waves. The Bianchi identities

$$\nabla_{[e} R_{ab]cd} = 0 \ \Leftrightarrow\ \nabla^d C_{abcd} = -\nabla_{[a}\Big\{R_{b]c} - \frac{1}{6} R g_{b]c}\Big\}, \tag{3}$$

are integrability conditions relating the Ricci tensor to the Weyl tensor [15, 18, 37], mediating the action at a distance of the gravitational field; they take a form similar to Maxwell's equations for the electromagnetic field [25]. Together with suitable equations of state for the matter, relating the various components of T_{ab}, equations (1), (2), (3) determine the dynamical evolution of the model and the matter in it.

I will simplify things in what follows by only considering the simplest form of matter: a pressure-free perfect fluid, such as cold dark matter or cold baryonic matter. This moves with a unique 4-velocity $u^a = dx^a/ds$ where s is proper time along the matter flow lines, so $u^a u_a = -1$. The matter energy-momentum tensor T_{ab} then takes the form

$$T_{ab} = \rho\, u_a\, u_b,\ \ \rho \geq 0\,, \tag{4}$$

where $\rho = T_{ab} u^a u^b$ is the energy density. The energy-momentum conservation equations (2) reduce to

$$\dot{\rho} = -\rho\, \nabla^a u_a\ ,\ \ \dot{u}_a := u^b \nabla_b u^a = 0\,, \tag{5}$$

so the matter, affected only by gravity and inertia, moves geodesically. This is the case of pure gravitation: it separates out the (non-linear) gravitational effects from the complexities of thermodynamic and fluid dynamical effects.

2 Timelike flows

For a given fundamental observer moving with 4-velocity u^a, spacetime decomposes into space and time [12, 15]. The metric of the tangent spaces orthogonal to u^a is given by

$$h_{ab} = g_{ab} + u_a u_b \ \Rightarrow\ \ h^a_c h^c_b = h^a_b,\ \ h^a{}_a = 3,\ \ h_{ab} u^b = 0 \tag{6}$$

where g_{ab} is the spacetime metric. The metric h_{ab} is used to project orthogonally to u^a; the volume element for tangent 3-spaces orthogonal to u^a is $\eta_{abc} = \eta_{abcd}u^d = \eta_{[abc]}$. Here round brackets denote symmetrization, square brackets denote skew symmetrization, and angle brackets represent the Projected Symmetric Trace-Free ('PSTF') part of a tensor. The covariant derivative of a geodesic timelike vector field u^a may be split into irreducible parts as

$$\nabla_b u_a = \frac{1}{3}\Theta h_{ab} + \eta_{abc}\omega^c + \sigma_{ab}\,, \quad \sigma_{ab} = \sigma_{<ab>}, \tag{7}$$

where

$$\Theta \equiv h^{ab}\nabla_a u_b\ , \quad \sigma_{ab} \equiv h^c_a h^d_b \nabla_{(c}u_{d)} - \frac{1}{3}\Theta h_{ab}\ , \quad \omega^a = \frac{1}{2}\eta^{abc}\nabla_{[c}u_{d]} \tag{8}$$

are the rate of expansion, shear, and vorticity respectively. Time and spatial derivatives relative to u^a for a tensor $T^a{}_b$ are defined by:

$$\dot{T}^a{}_b = u^c\nabla_c T^a{}_b\,, \ \tilde{\nabla}_c T^a{}_b = h_c{}^d h^a{}_e h_b{}^f \nabla_d T^e{}_f\,. \tag{9}$$

The Weyl tensor splits into the PSTF gravito-electric and gravito-magnetic fields

$$E_{ab} = C_{acbd}u^c u^d\ , \quad H_{ab} = \frac{1}{2}\eta_{acd}C^{cd}{}_{be}u^e\ , \tag{10}$$

which provide a covariant description of tidal forces and gravitational radiation.

The Einstein equation (1) with matter source term (4) and the Ricci identity

$$\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d \tag{11}$$

for u^a give the following evolution equations:

$$\begin{aligned}
\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a + \tfrac{1}{2}\kappa\rho - \Lambda &= 0\,, && (12)\\
\dot{\sigma}_{\langle ab\rangle} + \tfrac{2}{3}\Theta\sigma_{ab} + \sigma_{c\langle a}\sigma_{b\rangle}{}^c + \omega_{\langle a}\omega_{b\rangle} + E_{ab} &= 0\,, && (13)\\
\dot{\omega}_{\langle a\rangle} + \tfrac{2}{3}\Theta\omega_a - \sigma_{ab}\omega^b &= 0\,. && (14)
\end{aligned}$$

Constraint equations are the identity $\tilde{\nabla}_a\omega^a = 0$, the field equation

$$\tfrac{2}{3}\,\tilde{\nabla}^a\Theta - \tilde{\nabla}_b\sigma^{ab} - (\mathrm{curl}\,\omega)^a = 0, \tag{15}$$

where the 'curl' is $(\mathrm{curl}\,\omega)^a = \eta^{abc}\tilde{\nabla}_b\omega_c$, and an equation for the magnetic part of the Weyl tensor:

$$H^{ab} = -\tilde{\nabla}^{\langle a}\omega^{b\rangle} + (\mathrm{curl}\,\sigma)^{ab} \tag{16}$$

where the 'curl' is $(\mathrm{curl}\,\sigma)^{ab} = \eta^{cd\langle a}\,\tilde{\nabla}_c\sigma_d{}^{b\rangle}$. Propagation equations for the Weyl tensor are the $\dot{E}$-equation and $\dot{H}$-equations:

$$\begin{aligned}
\dot{E}^{\langle ab\rangle} - (\mathrm{curl}\,H)^{ab} &= -\tfrac{1}{2}\,\kappa\rho\,\sigma^{ab} - \Theta\,E^{ab} + 3\,\sigma^{\langle a}{}_c\,E^{b\rangle c} + \eta^{cd\langle a}\omega_c E_d{}^{b\rangle}\ , && (17)\\
\dot{H}^{\langle ab\rangle} + (\mathrm{curl}\,E)^{ab} &= -\Theta\,H^{ab} + 3\,\sigma^{\langle a}{}_c\,H^{b\rangle c} + \eta^{cd\langle a}\omega_c\,H_d{}^{b\rangle}\ , && (18)
\end{aligned}$$

where the 'curls' are $(\mathrm{curl}\,H)^{ab} = \eta^{cd\langle a}\,\tilde{\nabla}_c H_d{}^{b\rangle}$, $(\mathrm{curl}\,E)^{ab} = \eta^{cd\langle a}\,\tilde{\nabla}_c E_d{}^{b\rangle}$. The constraint equations are the $(\mathrm{div}\,E)$ and $(\mathrm{div}\,H)$-equations:

$$\begin{aligned}
\tilde{\nabla}_b E^{ab} &= \tfrac{1}{3}\,\kappa\tilde{\nabla}^a\rho + 3\,\omega_b\,H^{ab} + \eta^{abc}\,\sigma_{bd}\,H_c{}^d\ , && (19)\\
\tilde{\nabla}_b H^{ab} &= -\kappa\rho\,\omega^a + 3\,\omega_b\,E^{ab} - \sigma_{bd}\,E_c{}^d\ . && (20)
\end{aligned}$$

These equations - the exact non-linear equations for dust filled spacetimes - are clearly generalizations of Maxwell's equations for the electromagnetic field [25].

The case of a vacuum (empty spacetime) is a special case of these equations: just set $\rho = 0$. The general form of these equations for arbitrary matter fields (including pressure, viscosity, and heat flux terms) is given in [18].

2.1 Dynamic effects

By the vorticity conservation equation (14), the vorticity along each world line is affected only by the expansion and shear: for the case of pressure free matter considered here, vorticity cannot be created or destroyed along any world line. By the Raychaudhuri equation (12), the rate of expansion is increased by a cosmological constant but decreased by any matter present (because we have assumed the matter energy density is positive). This is the local Ricci effect on the fluid flow (it results from the Ricci tensor term in the EFE (1)), which occurs in a point-by-point manner due to the matter occurring along the worldline. The Weyl tensor cannot directly affect the expansion rate, but it can do so by inducing shear (via the shear propagation equation (13)) which then induces a deceleration (by (12)). This is the Weyl effect on the fluid flow; it occurs non-locally, due to matter at a distance from the world line. If $E_{ab} = 0$ the evolution equations along each world line (12) - (14) become ordinary differential equations unaffected by distant matter: there are no tidal effects or gravitational wave effects affecting the fluid flow, and each world line evolves on its own, unaffected by what is happening elsewhere. This is what has been called a 'silent universe' [26, 4] (for a recent review, see [42].)

How do non-local effects occur? We can think of it as happening in two ways. First, tidal action at a distance is represented by the the $(\operatorname{div} E)$ equation (19) with source the spatial gradient of the energy density (e.g. scalar perturbation modes); this can be regarded as a vector analogue of the Newtonian Poisson equation, where by matter elsewhere generates an E-field here. Similarly the $(\operatorname{div} H)$ equation (20) shows that fluid vorticity elsewhere generates a H-field here (e.g. vector perturbation modes).

Alternatively, we can consider the evolution equations (17), (18), which together form a hyperbolic system. They show how gravitational radiation arises: taking the time derivative of the $\dot{E}$-equation gives a term of the form $(\operatorname{curl} H)^{\cdot}$; commuting the derivatives and substituting from the $\dot{H}$-equation eliminates H, and results in a term in $\ddot{E}$ and a term of the form $(\operatorname{curl}\operatorname{curl} E)$, which together give the wave operator acting on E [22, 11]; similarly the time derivative of the $\dot{H}$-equation gives a wave equation for H. This shows how matter over there can effect matter here by generating gravitational radiation which travels here and causes a non-zero E-field here, which then affects matter here (via the shear equation (13). This is compatible with the constraint equation effects just discussed, because the constraint equations are preserved by the time evolution equations (see [40] corrected in [41]). As the key link between the E and H fields in this process is via their curls, this suggests one can characterize the existence of gravitational radiation by the condition

$$(\operatorname{curl} H)^{ab} \neq 0, \ (\operatorname{curl} E)^{ab} \neq 0 \tag{21}$$

which of course requires that both E and H are non-zero.

However there is another way a non-zero Weyl tensor can be created where there was none before: this is locally via matter shear (see (17)). This emphasizes the crucial

importance of shear in gravitational dynamics. It is not only the link whereby information on surrounding inhomogeneities (given us via the electric part of the Weyl tensor) alters the fluid flow here, it is also a source of the electric part of the Weyl tensor. If the shear is zero, this link is broken, the way distant matter can influence us here is very limited: for by (13), only the vorticity can prevent the electric part of the Weyl tensor from being zero. But as we see in the next section, this does not work; distant matter is constrained to acting in an isotropic way around our world line. This is a very special situation.

3 Timelike shear-free results

When the shear is zero, the expansion is isotropic; we might expect vorticity to tend to generate anisotropy that would break this condition. However a non-zero Weyl tensor might balance this tendency. Specifically, on setting $\sigma_{ab} = 0$ in the above equations,(13) becomes a new constraint equation, along with the old constraint (16) determining E and H in terms of ω:

$$E_{ab} = -\omega_{\langle a}\omega_{b\rangle}, \;\; H^{ab} = -\tilde{\nabla}^{\langle a}\omega^{b\rangle} \; . \tag{22}$$

The task now is to take take time derivatives of these constraints to see if the shear-free equations (obtained by setting $\sigma_{ab} = 0$ in all the above equations) are consistent for some non-trivial special cases.

This is a major calculation: the result is not obvious. It was not initially tackled this way. First, in a remarkable pioneering paper presented at a world mathematics congress in 1950, Kurt Gödel examined this question in the case of spatially homogeneous Bianchi IX cosmologies. He showed [19] that in this case, a shear-free universe could either expand or rotate, but not both; but he did not show how he had obtained that result. In 1957, Schüucking derived the Gödel result in detail [35]. In 1967, I used an orthonormal tetrad formalism to show that the restriction of spatial homogeneity was unnecessary:

> **Dust Shear-Free Theorem** [16]: if a dust solution of the EFE (possibly with a cosmological constant) is shearfree in a domain U, it cannot both expand and rotate in U:
> $$\{\dot{u}^a = 0, \; \sigma_{ab} = 0\} \Rightarrow \omega\Theta = 0. \tag{23}$$

This is an exact result, obtained by utilizing all the field equations. A covariant proof is given in [36]. Applying this theorem to the cosmological context, consider a shear-free dust-filled universe that expands. Then (23) shows $\omega_{ab} = 0$, so from the above equations

$$\{\sigma_{ab} = 0, \; \Theta > 0\} \Rightarrow E_{ab} = 0, \; H^{ab} = 0, \; \tilde{\nabla}^a\Theta = 0. \tag{24}$$

The space-time is conformally flat and the universe is a Friedmann-Lemaître-Robertson-Walker universe [15]. These are thus the only expanding shear-free baryonic plus CDM cosmological solutions, provided both these components move with the same 4-velocity.

One should note that the result (24) does not require $\rho > 0$. It is true in the vacuum case (with or without a cosmological constant). Later generalizations considered perfect fluids rather than pressure-free matter, so acceleration of the timelike congruence was allowed; the result (23) remains true in all cases considered so far, indeed Collins

conjectured [8] that all shear-free perfect fluids obeying a barotropic equation of state must have either zero expansion or zero vorticity. Senovilla, Sopuerta and Szekeres [36] summarized results obtained towards proving this conjecture, and gave a fully covariant proof that shear free solutions with the acceleration vector proportional to the vorticity vector (including the case of vanishing shear) must be either non-expanding or non rotating. Van den Bergh [38] gave a tetrad-based approach for two particular cases require a special treatment, namely $p + 1/3\rho =$ constant, and $p - 1/9\rho =$ constant, as well as the equation of state $p = (\gamma - 1)\rho + constant$. Van Den Bergh, Carminati, et al [39, 5] showed the result is generically true for shear-free perfect fluid solutions of the Einstein field equations where the fluid pressure satisfies a barotropic equation of state and the spatial divergence of the magnetic part of the Weyl tensor is zero.

Can one get models other than FLRW in these cases? Collins showed [7] that for irrotational shearfree perfect fluids obeying a barotropic equation of state $p = p(\mu)$ and with nonzero acceleration, one can get spherically symmetric Wyman solutions, or models that are plane symmetric, and either spatially or temporally homogeneous. In all cases, when the space-time is sufficiently extended, the fluid exhibits unphysical properties. Consequently shear-free expanding barotropic perfect fluids must either be FLRW, or must be restricted to local regions where these conditions hold. Thus it turns out that the FRW models are the only shear-free barotropic perfect fluid models in which the matter is physically reasonable globally [8].

Overall, these results show clearly how restrictive the shear-free result is for plausible fluid models. It will of course not be true for "imperfect fluids" with arbitrary equations of state: one can then just run the field equations from left to right to determine an unphysical form of "matter" that will give the desired result. Such calculations have no physical significance. One should note here that despite what one might have thought at first, although shear-free solutions are necessarily self-similar (they map the orthogonal 3-spaces conformally onto each other), the converse is not necessarily true: self similar solutions need not be shear-free (see e.g. [6]).

This is related to the idea of perfect fluid as follows: a continuum description of matter is underlain by a kinetic theory description. Now it follows from kinetic theory for a collision-free fluid that if there is non-zero shear, there will be an anisotropic stress (the non-zero shear will generate anisotropy in the particle distribution function which will then result in an anisotropic pressure [17]), hence a perfect fluid description will not then apply. This will also be true if there are collisions leading to a non-zero shear viscosity, because of the relation

$$\pi_{ab} = \lambda\, \sigma_{ab} \tag{25}$$

determining the anisotropic pressure π_{ab}, where λ is the shear viscosity coefficient [12, 15]. It follows that if one has a perfect fluid,

$$\{\pi_{ab} = 0, \lambda \neq 0\} \Rightarrow \{\sigma_{ab} = 0\}. \tag{26}$$

But kinetic theory shows that $\lambda \neq 0$ for realistic fluid descriptions based on kinetic theory with collisions [24, 17]. Hence *an exact perfect fluid description of continuous matter implies zero shear*, and the above results apply.

The moral is that realistic matter cannot be accurately represented by a perfect fluid description. The difference may be small, but is important in principle.

3.1 The Newtonian limit

This shearfree result is an exact result for the full non-linear EFE: no approximations have been made, and is completely general: it holds for any spatially homogeneous or inhomogeneous model. This raises a very interesting situation as regards the Newtonian limit of the EFE, because this result is not true in the Newtonian case.

The key point here is that Newtonian Gravity is not independent of General Relativity: it derives from General Relativity in special conditions. Specifically, it is a limiting form of General Relativity, valid in particular circumstances (when matter relative motion is at low speeds, and there are no gravi-magnetic effects or gravitational waves, which will be true if the magnetic part of the Weyl tensor is zero). Consequently the properties of Newtonian gravity should follow from those of general relativity.[1] One should note here that to obtain Newtonian cosmological models, one has to use a description in terms of a gravitational potential where one allows the potential to diverge at infinity, so it is not strictly Newtonian theory, but rather an extension of the theory, where also the idea of acceleration is generalized. With these generalizations, one can obtain viable Newtonian cosmological models for the dynamical behavior in a matter dominated era[2] (the relevant equations and references are given in [15]).

The major point then is that there are shearfree solutions of the Newtonian equations for pressure-free matter in cosmology that both expand and rotate; specific examples have been given by Narlikar [27]. Consequently, the Newtonian limit is singular. Consider a sequence $GRT(i)_{\sigma=0}$ of relativistic shearfree dust solutions with a limiting solution $GRT(0)_{\sigma=0}$ that constitutes the Newtonian limit of this sequence. The latter solution will necessarily satisfy (23) because every solution $GRT(i)_{\sigma=0}$ in the sequence does so. Newtonian solutions $NGT_{\sigma=0}$ that do not satisfy (23) are thus not obtainable as limits of any sequence of relativistic solutions $GRT(j)_{\sigma=0}$. Assuming Einstein's field equations represent the genuine theory of gravitational interactions in the physical Universe, this result tells us that not all Newtonian cosmological solutions are acceptable approximations to the true theory of gravity.

An important application of this result is as follows: Narlikar has shown [27] that shearfree and expanding Newtonian cosmological solutions can have vorticity that spins up as the universe decreases in size, and hence causes a 'bounce' (the associated centrifugal forces avoid a singularity). This would be a counter-example to the cosmological singularity theorems of Hawking and Penrose (see [23]), if there were GRT analogues of these singularity-free cosmological solutions; but the shearfree result (23) shows there are no such GR solutions. This is a remarkable way in which the exact properties of GR dust solutions support the results of the Hawking-Penrose singularity theorems, obtained by completely different methods.

This is a case where the Newtonian models are very misleading. The Newtonian limit is singular in such cases; so we need to be cautious about that limit in other situations of astrophysical and cosmological interest.

[1] This is in contrast to the view of some astrophysicists that Newtonian theory is the true gravitational theory, and GRT a set of small corrections to be made to this true theory.

[2] This is not possible for a radiation dominated era.

3.2 The linearised case

It is of considerable interest then whether the result (23) holds in the case of linearised perturbations of FLRW universe models. It has recently been shown[3] that it holds in this case too: if a perfect fluid with equation of state $p = k\rho$ in an almost FLRW universe is shear-free, then it must be either expansion-free or rotation-free. Thus linearization does not lose this property. This makes the situation even more remarkable: for then the linearised solutions - almost universally used to study the formation of structure by gravitational instability in the expanding universe, and believed to result in standard local Newtonian theory - also does not give the usual behaviour of Newtonian solutions in interesting cases. The moral of the story is that you can't believe a Newtonian solution unless it can be shown to be the limit of a family of GRT solutions - else it may lead you badly astray (as in the case of the shear-free expanding and rotating solutions found by Narlikar).

4 Null flows

I now turn to the null case. The kinematic definitions of expansion, shear and vorticity for congruences of null geodesics were given by Ehlers and Sachs [14], with a tetrad version being given by Newman and Penrose [28] (summaries are given in [23], [31]). This proceeds in parallel to the analysis for the timelike case, with two crucial differences: the geodesic vector field $k^a = dx^a/dv$ is null ($k^a k_a = 0$) rather than timelike; and the projection is into a two-dimensional spacelike screen space orthogonal to k^a and an observer u^a, instead of into a 3-dimensional space as in the case of u^a (so in this section, $< .. >$ denotes trace-free 2-dimensional orthogonal projection to k^a and u^a).

For an irrotational null geodesic congruence, the optical scalars $\hat{\theta}$ (expansion, given by $\nabla_a k^a = 2\hat{\theta}$) and $\hat{\sigma}_{ab} = \hat{\sigma}_{<ab>}$ (shear) satisfy the *Sachs equations*

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}v} + \hat{\theta}^2 + |\hat{\sigma}|^2 = \Phi_{00}, \tag{27}$$

$$\frac{\mathrm{d}\hat{\sigma}_{ab}}{\mathrm{d}v} + 2\hat{\theta}\,\hat{\sigma}_{ab} = \Psi_{ab} \tag{28}$$

where the Ricci tensor term is $\Phi_{00} := \kappa(\rho + p)$, determined by the matter at each point, and the Weyl tensor term is $\Psi_{ab} := k^c C_{c<ab>d} k^d$, determined by matter elsewhere plus boundary conditions.

These are obviously in analogy to the timelike case (12), (13) (there is no analogue to (14) because the vector field k^a is a gradient: $k_a = \nabla_a \phi$, and so is irrotational). Thus one can again refer to *Ricci focusing* caused pointwise by the matter distribution Φ_{00} inside the beam,[4] and *Weyl focusing* caused by the Weyl tensor term Ψ_{ab} generated non-locally by matter outside the beam. It is the latter effect that underlies gravitational lensing and consequent focussing of the null geodesic rays. The full set of equations whereby these non-local effects take place are given in terms of the spin coefficient equations in ([30], pp.248-249); these are equations (12) - (20) above expressed in terms of a null vector

[3] A-M Nzoiki, R Goswami, P K S Dunsby, and G F R Ellis, in preparation.

[4] It is noteworthy that the cosmological constant does not enter here: it has no direct influence on null focusing.

basis, plus equations determining the tetrad rotation coefficients and giving the tetrad components relative to a coordinate basis (required to get a complete set of equations).

These equations immediately imply

$$\{\Phi_{00} = 0, \hat{\theta} = 0\} \Rightarrow \hat{\sigma} = 0 \tag{29}$$

i.e. an irrotational null congruence in empty space cannot shear without either expanding or contracting. If we had included twist, the conclusion would have been altered to, an irrotational null congruence in empty space cannot shear without either expanding/contracting or twisting: a kind of inverse of the timelike no-shear result.[5]

4.1 The geometry of the Weyl tensor

The Petrov classification describes the possible algebraic properties of the Weyl tensor at each event in a Lorentzian manifold. It was first given in terms of an orthonormal basis by Petrov [32], and is nicely described by Ehlers and Kundt [13] and by Penrose and Rindler ([31], pp. 242-246).

The relation to null vectors was given by Ehlers and Sachs [14], showing how the timelike and spacelike Weyl eigenbivectors are associated with preferred null vectors, called the *principal null directions* (PND's) of the Weyl tensor. The condition for k^a to be a principal null direction of the Weyl tensor is

$$k_{[e}C_{c]ba[d}k_{f]}k^c k^d = 0. \tag{30}$$

In general, there are four uniquely determined PND's at each point if the Weyl tensor is non-zero, but in degenerate cases two or more PND's may coincide; the degenerate PND's satisfy more restrictive equations than (30): k^a is a degenerate eigendirection of the Weyl tensor iff

$$C_{abc[d}k_{e]}k^b k^c = 0. \tag{31}$$

The different possibilities lead to the six Petrov types, succinctly described using a spinor formalism ([29]; [31], pp. 223-226); these different algebraic types correspond to different physical situations. The Petrov types are,

- Type I : four simple PNDs (generic: realistic models),
- Type II : one double and two simple PNDs,
- Type D : two double principal null directions (massive objects with symmetry: e.g, Schwarzschild),
- Type III: one triple and one simple PND,
- Type N : one quadruple PND: $C_{abcd}k^d = 0$ (pure gravitational waves: e.g. a plane gravitational wave),
- Type O: the Weyl tensor vanishes: $C_{abcd} = 0$ (no tidal forces: e.g. a FLRW universe).

Type I is generic, all the others are algebraically special Weyl types (they all have repeated PNDs) and so correspond to restricted physical situations with symmetries.

[5] There is no corresponding timelike result if the cosmological constant is non-zero.

4.2 The Goldberg Sachs Theorem

As in the timelike case, for essentially the same reasons, existence of shear-free null congruences (discussed in [31], pp. 189-199)[6] places strong restriction on the spacetime. We immediately find

$$\begin{aligned} \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}v} + \hat{\theta}^2 &= \Phi_{00}, &(32)\\ 0 &= \Psi_{ab}, &(33)\end{aligned}$$

the first showing that no new gravitational information can enter the congruence as it travels from the source to the observer (only the matter encountered by the null rays can cause convergence) and the second shows that shear-free null geodesics are PNDs ([31], (7.2.14) and (7.3.2)). In the vacuum case, further restrictions occur, captured in an important result by Joshua Goldberg and Rainer Sachs:

> **Goldberg-Sachs Theorem** [20]: A vacuum metric admits a shear free null geodesic congruence k^a if and only if k^a is a degenerate eigendirection of the Weyl tensor (equation (31) is true).

This proves that a vacuum solution of the Einstein field equations will admit a shear-free null geodesic congruence if and only if the Weyl tensor is algebraically special. Shortly after the Goldberg and Sachs paper, an alternative proof was given by Newman and Penrose [28], using a tetrad or spinor formalism. A generalisation was given by Robinson and Schild [33], establishing a connection between algebraic degeneracy of the Weyl tensor, the existence of a null geodesic shear-free congruence, and restrictions on the Ricci tensor which are weaker than the requirement that there be empty space. in particular, they showed that the gravitational field due to any Maxwell field with shear-free rays is algebraically special. An even more generalized version is given by Penrose and Rindler ([31], pp. 195-198).

The theorem is useful in searching for algebraically special vacuum solutions, which is very helpful because almost all solutions we can write down in exact form are algebraically special, corresponding to restricted matter distributions and boundary conditions; examples are the Kerr solution and plane gravitational waves.

4.3 The News function

Given the discussion above, one might expect that shear relates to the way information is conveyed along bundles of null geodesics, with restrictions on that information when the shear is zero. This expectation seems to be borne out by the analysis of axisymmetric vacuum spacetimes by Bondi, van den Berg and Metzner [3], conveniently summarised in the book by D'Inverno [10]. On using a null coordinate system where the past null cones of the central observer are given by $\{u = const\}$ where u is the retarded time and r is a measure of distance down the past light cone, the metric is the Bondi metric

$$ds^2 = -(\frac{V}{r}e^{2\beta} - r^2 e^{2\gamma} U^2)du^2 + 2e^{2\beta}dudr + 2Ur^2 e^{2\gamma}dud\theta - r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2) \quad (34)$$

[6] Shear free null congruences play an important role in twistor geometry ([31], Chapter 7).

where $V = V(u,r,\theta)$, $U = U(u,r,\theta)$, $\beta = V(u,r,\theta)$, $\gamma = \gamma(u,r,\theta)$. Then solving the vacuum EFE asymptotically, $V = r - 2M + O(r^{-1})$, $\beta = -n^2/4r^2 + O(r^{-3})$, $4q_{,u} = 2Mn - d_{,\theta} + d\cot\theta$, and

$$\gamma = \frac{n(u,\theta)}{r} + \frac{q(u,\theta)}{r^3} + O(r^{-4}). \tag{35}$$

The mass of the system as measured at infinity is the Bondi Mass

$$m(u) = \frac{1}{2}\int M(u,\theta)\sin\theta d\theta \tag{36}$$

The shear of the radially outgoing null rays is

$$\hat{\sigma} = n(u,\theta)/r^2. \tag{37}$$

The initial data is $\gamma(u,r,\theta)$ on an initial value null surface, plus $n_{,u}(u,\theta)$ which determines the evolution of the source and so is called the *News Function*, determined by the first time derivative of the shear. Finally

$$m_{,0} = -\frac{1}{2}\int_0^{\pi}(n_{,0})^2\sin\theta d\theta \tag{38}$$

which is non-positive and so shows that *there is mass loss if and only if there is news.*

The Weyl tensor up to order $1/r$ has the non-zero outgoing radiation component

$$\Psi_0 \;\; = \;\; -n_{,uu}/r = -r\,\hat{\sigma}_{,uu} \tag{39}$$

so Bondi's formula says that the Bondi mass of the system decreases if and only if there is outgoing radiation, and so if and only if the second time derivative of the shear is non-zero. As stated in the abstract, "*It is shown that the flow of information to infinity is controlled by a single function of two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines the behaviour of the system..... The principal result of the paper is that the mass of a system is constant if and only if there is no news; if there is news, the mass decreases monotonically so long as it continues.*" This confirms the key role played by shear in conveying news, and hence in the mass loss carried out by outgoing gravitational radiation. The result remains true if the assumption of axisymmetry is dropped [34].

Thus we have confirmation of the key concept of this paper: the shear of geodesics, generated by the Weyl tensor, conveys information about the gravitational field due to gravitating bodies. If the shear is zero, this information is very limited and the space time dynamics is highly constrained.

4.4 The discontinuous limit

However there is an intriguing context where this picture seems to fail. The outgoing gravitational waves discussed in the previous section should at large distances asymptotically become plane gravitational waves.

The pp wave geometries [1] are given in terms of null coordinates by

$$ds^2 = -2dudv + h_{ij}(x,u)dx^i dx^j \tag{40}$$

with a covariantly constant null vector field $\xi^a = (\partial/\partial u)^a$ which is consequently a Killing vector whose expansion and shear vanish. Hence they are subset of null shear-free solutions; the Weyl tensor is type N and this vector field is a 4-fold degenerate PND (thus satisfying the Goldberg-Sachs theorem). Plane gravitational waves [2] are a special class of vacuum pp-wave where

$$ds^2 = -2dudv + [a(u)(x^2 - y^2) + 2b(u)xy]du^2 + dx^2 + dy^2 \tag{41}$$

Here, $a(u)$ and $b(u)$ can be any smooth functions; they specify the amplitude of the two polarization modes of gravitational radiation. The waves can convey arbitrary messages by the time variation of these modes along successive light cones.

The issue then is the following:

1: The Bondi news function for asymptotically flat vacuum metrics is the time derivative of the shear. Hence, no shear implies no news.

2: The BMS metrics (34) should become plane gravitational waves asymptotically at infinity.

3: The pnd vector field of a plane gravitational wave is necessarily an exactly shear-free geodesic congruence (Goldberg-Sachs).

4: But plane fronted gravitational waves can freely carry news: they have two free polarisation functions, even though their shear is exactly zero.

5: Hence the Bondi metrics should tend to a state where there is no shear at infinity, hence no news, yet effective news transfer is possible in the exact limiting spacetime.

How does this all fit together? Presumably it is a question of relative orders of decay near infinity: but it is not obvious how it is coherent! One might have thought that zero shear meant zero news: but this is not the case!

There is of course a shear associated with plane waves, which is essential for the existence of the waves. Null geodesics which intersect the null hyperplane histories of the plane waves must have shear (if they don't then the plane waves don't exist). Other manifestations of this phenomenon are Penrose's observation that an impulsive plane gravitational wave acts as an astigmatic lens and also that colliding plane waves generate shear after collision. But the paradox remains: the PND congruence is shear free, and indeed as a consequence the geometry is very limited (it has a large symmetry group) but still can convey arbitrary information.

4.5 Linearised gravity

There is no Newtonian analogue of the Goldberg-Sachs theorem, as there are no equivalent concepts there. But one can consider the linearized version of the result. It has been shown by Dain and Moreschi [9] that a corresponding theorem will not hold in linearized gravity, that is, given a solution of the linearised Einstein field equations admitting a shear-free null congruence, then this solution need not be algebraically special.

This is a warning of the perils of using linearised results for a non-linear theory: some key results may not be valid in the linearised case, even though they are an exact result of the non-linear theory. Hence as in the case of the Newtonian limit of the timelike case, here we can consider a sequence of exact shearfree solutions that tend to a linearised shear free solution: the limiting properties of the exact solutions differ from the properties of

the linearised solution. Hence for example special solutions of the exact and linearised equations may have differen properties.

5 Conclusion

Shear of geodesic curves plays a crucial role in conveying information regarding the state of matter in a region, underlying structure formation in the timelike case and gravitational lensing in the null case.

I have compared the timelike and null cases of shear-free geodesics. In each case the allowed exact solutions are strongly restricted, albeit in rather different ways. The argument makes clear that the two contexts are related in the following way: one would not expect gravitational radiation to be emitted by a shear-free dust flow, and that is indeed the case (combine (24) and (21)). We have also seen that there are intriguing limiting questions in each case: shear-free solutions don't have the same implications in Newtonian theory (timelike case) and linearised gravity (the null case). Pursuing these issues might be interesting.

One further line of of investigation that may also be interesting is to pursue the stability of the Goldberg-Sachs result in the following sense: an "almost Birkhoff theorem" has recently been proven [21]. This shows that if the conditions for Birkhoff's theorem are almost true, then Birkhoff's theorem will be approximately satisfied. An "Almost Goldberg-Sachs theorem" would similarly show that if the conditions for the Goldberg-Sachs theorem are almost true, then Goldberg-Sachs theorem will be approximately satisfied. If this were not true, that might throw light on the failure of the asymptotic limit discussed in the previous section.

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